Some Computations with Seiberg–Witten Invariant Actions

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We show, with a 2–dimensional example, that the low energy effective action which describes the physics of a single D–brane is compatible with T–duality whenever the corresponding $U(N)$ non–abelian action is form–invariant under the non–commutative Seiberg–Witten transformations. [1]

1 Introduction and Basic Setting

It has been shown, by Seiberg and Witten [1], that, in the presence of a background magnetic field, the dynamics on D–branes can be described in terms gauge fields on a non–commutative space. Most of the subsequent literature has been devoted to the study of the low–energy $\alpha' \to 0$ limit, which yields a non–commutative Yang–Mills quantum field theory. In [3] the author has considered, on the other hand, the full effective action, to all orders in α', which is considered as a *classical* action which describes, at tree level, the scattering of gluons at weak coupling but arbitrarily high energies. As described in the work of Seiberg and Witten, in the presence of a background B_{ab} field, one can describe the physics on the brane either using ordinary gauge fields or, after an appropriate field redefinition, using non–commutative gauge fields. The form of the action, on the other hand, is the same in the two descriptions. This fact is quite non–trivial, and highly constrains the form of the non–abelian Born–Infeld action, together with all the derivative corrections.

In particular, in [3], the author has shown how to construct explicitly to all orders in α', invariant non–abelian actions in 2 dimensions. We show in this note how these actions, in the special $U(1)$ abelian case, are compatible with T–duality. This shows that Seiberg–Witten invariance is the correct framework to use in the analysis of the physics on brane world–volumes. In particular, this paper represents a partial step towards an explicitly covariant description of the brane world–volume action, whose invariance group in enhanced from the usual geometrical invariance of diffeomorphisms on the world–volume to a more exotic invariance, of which we still lack a clear geometric understanding.

Let us then consider a flat space time, and let us look at the physics of N flat D–branes with a 2–dimensional world volume M. We let x and y be the coordinates on the world–volume M, and we will concentrate only on the physics in the x–y plane, neglecting the dynamics in the transverse directions in space–time. Finally, we will use throughout a metric δ_{ab} on M with *Euclidean* signature and we will use units such that $2\pi\alpha' = 1$.

The physics on the brane world–volume, at weak string coupling, is described by an effective action written in terms of a $U(N)$ gauge–potential A_a (with corresponding field–strength F_{ab}), which is given as an α' expansion

$$S = \int d^2x \, \mathrm{Tr}\left(1 + \frac{1}{4}F^2 + \cdots\right). \tag{1}$$

Let us, for the moment, consider the case $N = 1$, and let us perform a T–duality in the y direction. Following basic facts on T–duality [2], we are now considering the physics of a D–brane with 1–dimensional world–volume, moving in the x–y plane. The action then depends on the

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motion of a point on the Euclidean plane M – *i.e.* a curve Γ in M. More precisely, following [2], one must consider the action S, dimensionally reduced in the y direction, where we may consider $A_x = 0$ and where

$$A_y(x) \to Y(x)$$

represents a specific parametrization of the curve Γ in the plane M. Let us call $\widetilde{S}$ the $N = 1$ dimensionally reduced action, coming from S.

When we move from S to $\widetilde{S}$, we obtain an action written in terms of a specific parametrization $Y(x)$ of the path Γ in the plane M. On the other hand, from a purely geometric point of view, the action must be invariant under reparametrizations of Γ. In particular, given the simplicity of the geometry, it is easy to describe all possible invariant actions $\widetilde{S}$. First of all, let us define the velocity and acceleration

$$V = \dot{Y} \qquad\qquad A = \ddot{Y},$$

where the dot represents differentiation with respect to x. The invariant length dl on the path Γ and the extrinsic curvature R are respectively given by

$$dl = \left(1+V^2\right)^{\frac{1}{2}}\, dx \qquad\qquad R = \frac{A^2}{\left(1+V^2\right)^3}.$$

Finally, covariant differentiation with respect to arc–length is given by

$$\frac{d}{dl} = \left(1+V^2\right)^{-\frac{1}{2}} \frac{d}{dx}.$$

The possible terms which appear in the action are then integrals, with respect to arc–length dl, of powers of R and of its covariant derivatives $\frac{d^n R}{dl^n}$. Examples of possible terms are

$$\int dl \qquad\qquad \int dl\, R^n \qquad\qquad \int dl\, R^n \frac{dR}{dl}\frac{dR}{dl}. \tag{2}$$

The description of the possible structures of $\widetilde{S}$ just given is extremely natural (and in this setting almost trivial). On the other hand, from the point of view of the original 2–dimensional action S, written in terms of the $U(1)$ gauge potential A_a, the fact that one obtains, after T–duality, actions written in terms of monomials like (2), is a quite remarkable numerical coincidence. Let us, for instance, consider the first term in (2). As it is well known, this corresponds, from the 2–dimensional point of view, to the abelian Born–Infeld action. We will denote, in what follows, with $F \equiv F_{xy}$ the unique component of the field–strength. Under T–duality this corresponds to the velocity

$$F \to_T V.$$

Therefore, the length of the path Γ is given by

$$\int dl = \int dx \left(1+V^2\right)^{\frac{1}{2}} \to \int d^2x \left(1+F^2\right)^{\frac{1}{2}} = \int d^2x \left(1 + \frac{1}{2}F^2 - \frac{1}{8}F^4 + \cdots\right).$$

The fact that the LHS of the above equality has a geometric interpretation imposes restrictions on the α' expansion coefficients which appear on the RHS of the equality. More generally, the α' expansion of the low–energy effective action S is constrained by compatibility with the geometric interpretation of T–duality.

Quite independently from the above considerations, the action S possesses [1], in the full non–abelian setting, an other interesting invariance property. If one considers adding a central term to the field–strength $F_{ab} \to F_{ab} + \mathbf{1} \cdot B_{ab}$, this has the same effect as rewriting the same action on a new world–volume $\widetilde{M}$, which is a non–commutative deformation of the original Euclidean plane

M, given by the relation $[x,y]=i\theta^{xy}$. The tensor θ is given by $\theta=B\left(1+B^2\right)^{-1}$. Moreover, the metric on the new plane $\widetilde{M}$ is given by $\left(1+B^2\right)$. The new action is now written in terms of a non–commutative Yang–Mills potential (explicitly given in terms of the original A_a and θ^{ab}) and has the *same structure* as S. This fact imposes severe constraints on the original α' expansion given in (1), as was analyzed in detail in [3], and in particular relates terms of order $(\alpha')^L$ to those of order $(\alpha')^{L-1}$ and $(\alpha')^{L+1}$. We will call the above restriction Seiberg–Witten (SW) invariance of S.

We will show in this note that the SW invariance of S in the non–abelian setting implies, in a quite non–trivial way, consistency under T–duality of the $U(1)$ action, as discussed above.

2 Invariant Actions

In this section we review how to construct SW invariant action in 2–dimensions. We will describe the algorithm, but we will not try to motivate it, since this would imply a long digression. The interested reader can consult [3], where the full SW invariance is discussed in detail.

First we introduce on M complex coordinates $z,\overline{z}=\frac{1}{\sqrt{2}}(x\pm iy)$. Corresponding to these coordinates, we introduce *formal letters* $Z,\overline{Z}$, and we consider the vector space W_L of *cyclic words* built with L letters Z and L letters $\overline{Z}$. Vectors in W_L are then linear combinations of strings of $2L$ letters, where we consider two strings equal if they differ only by a cyclic permutation of the letters. We will call the integer L the level, and this will correspond, as we will see below, to the level in the α' expansion (with the F^2 term at level 2).

We will actually be more interested in the subspace $G_L\subset W_L$ of so–called *gauge invariant words*, which are defined as follows. Let $a,a^\dagger$ be standard creation and annihilation operators satisfying $\left[a,a^\dagger\right]=1$ and let O be the space of cyclic polynomials in $a,a^\dagger$. If we define the map $r:W_L\to O$ by $Z,\overline{Z}\mapsto a,a^\dagger$, then $G_L=\ker r$. Elements in G_L correspond to terms in S at level $(\alpha')^L$. More precisely, we have the correspondences

$$\begin{aligned} F &\;\to\; Z\overline{Z}-\overline{Z}Z \\ D_z\cdots &\;\to\; i\left[Z,\cdots\right] \qquad\qquad D_{\overline{z}}\cdots\to i\left[\overline{Z},\cdots\right] \end{aligned} \tag{3}$$

where D_a is the $U(N)$ covariant derivative. For example, one can check that $\dim G_2=1$, and that the unique element is given by $Z\overline{Z}Z\overline{Z}-ZZ\overline{Z}\,\overline{Z}$, which, following the above identifications, corresponds to the term $\frac{1}{2}\int d^2x\,\mathrm{Tr}F^2$ at level 2. It is also clear that, given any expression in S, one can use the identifications (3) to rewrite it in terms of words in $Z,\overline{Z}$. The words will automatically be cyclic, since the action S is given by taking the "trace" $\int d^2x\,\mathrm{Tr}$ of a local expression in the field strength. Moreover, the original gauge invariance of the action is reflected in the fact that the resulting words are elements of the subspace G_L.

To construct SW–invariant actions we need to introduce two operators – denoted $\Delta,\overline{\Delta}$ – which act naturally on the spaces G_L as

$$\Delta:G_L\to G_{L+1}\qquad\qquad\qquad \overline{\Delta}:G_L\to G_{L-1}.$$

In particular, the operator Δ is defined as a derivation which acts on the single letters as

$$\Delta Z=\frac{1}{4}(ZF+FZ)\qquad\qquad\qquad \Delta\overline{Z}=\frac{1}{4}(\overline{Z}F+F\overline{Z}).$$

Similarly $\overline{\Delta}$ is defined by

$$\overline{\Delta}F=1.$$

More precisely, given a word $w \in G_L$, we write it in terms of the field–strength F (and of its covariant derivatives) using the correspondence (3). We then apply $\overline{\Delta}$ on each F as a derivation. It is quite a non–trivial fact [3] that $[\overline{\Delta}, \Delta] = 2L - 1$.

With this notation in place, we can now easily describe SW–invariant actions, following the results of [3]. A general invariant action is given by arbitrary linear combinations of what we will call *invariant blocks*. An invariant block is itself constructed starting from a *lowest level term* $g_P \in G_P$, for some integer P, and contains terms in the α' expansion of levels $L \geq P$, with $L - P \in 2\mathbf{N}$. The term g_P must satisfy the basic equation

$$\overline{\Delta} g_P = 0,$$

and the invariant block is then constructed as follows. We construct terms $g_L \in G_L$ for $L > P$ as

$$g_{L+1} = \Delta g_L.$$

It follows from $\left[\overline{\Delta}, \Delta\right] = 2L - 1$ that $\overline{\Delta} g_{L+1} = c_{L,P} g_L$, where

$$c_{L,P} = (L + P - 1)(L - P + 1).$$

Construct then coefficients d_L for $L - P \in 2\mathbf{N}$ defined by $d_P = 1$ and by the recursion relation

$$d_{L+2} = -\frac{d_L}{c_{L+1,P}}.$$

The invariant block constructed from g_P is then given by

$$\sum_{L-P \in 2\mathbf{N}} d_L \, g_L.$$

3 Abelian Actions

Up to now, we have discussed how to construct general $U(N)$ actions which are SW invariant. We now wish to analyze in more detail the invariant actions in the abelian $N = 1$ case. As is well known, in this case one can classify terms in the action S not only by the level L in the α' expansion, but also by the number D of derivatives appearing in the various terms. For example, a term like $F^2 \partial F \partial F$ has $L = 5$ and $D = 2$. It is not difficult to show from the definitions (see [3]) that the operation Δ increases L by 1 and *does not decrease* D. On the other hand, the operator $\overline{\Delta}$ decreases L by one unit and *leaves D unchanged*.

Let us now consider an invariant block with a lowest term g_P with level P and number of derivatives D. We claim that, with very little computation, we can resum all the terms in the invariant block with exactly D derivatives. To show this, first recall that $\overline{\Delta}^{L-P} g_L \propto g_P$. Since g_P contains D derivatives, given the properties of $\Delta, \overline{\Delta}$ one has

$$\overline{\Delta}^{L-P} g_L \Big|_{D\text{derivatives}} \propto g_P.$$

Moreover, since the action of $\overline{\Delta}$ on terms with D derivatives is completely known, one can quickly reconstruct the full invariant block. We will give below a concrete example, which will clarify the general discussion.

We will show that the D derivative part of an invariant block is explicitly given by the covariant terms described in section 1 (as in equation (2)). A generic covariant term is constructed with N_R powers of curvature R and with N_D covariant derivatives $\frac{d}{dl}$. Recalling the correspondences $V \to F$, $A \to \partial F$, $\cdots$ it is simple to show that

$$P = 3N_R + \frac{1}{2} N_D \qquad\qquad D = 2N_R + N_D.$$

Since $N_R, N_D \geq 0$, we then have that

$$\frac{1}{2}D \leq P \leq \frac{3}{2}D.$$

In Figure 1 we show a graph of the possible P, D combinations for lowest level terms, and we show the corresponding covariant terms.

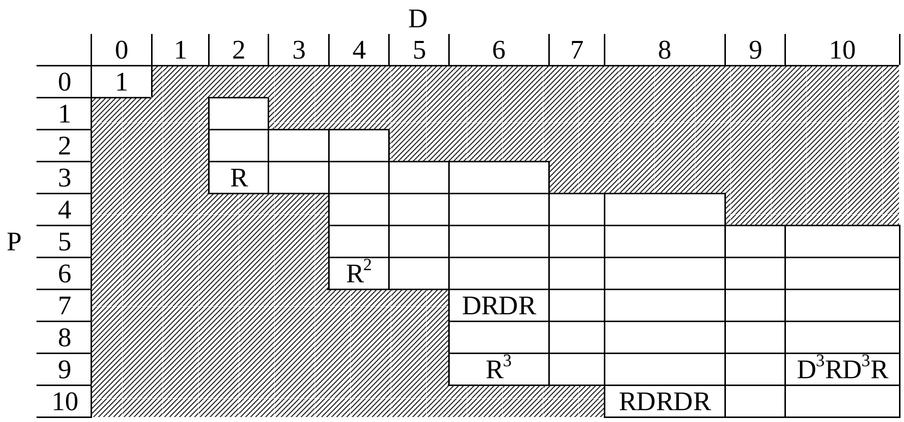

Figure 1. Derivative with respect to arc–length dl is denoted by D in the figure.

4 Examples

Let us consider the covariant term

$$\int dl\, R^n = \int dx \frac{A^{2n}}{\left(1+V^2\right)^{3n-\frac{1}{2}}} \tag{4}$$

We clearly have that $P = 3n$ and $D = 2n$. Explicitly one has (we are using the T–duality correspondences $F \to V$, $\cdots$)

$$\begin{aligned} g_P &= A^{2n} & g_{P+1} = (2P-1)\, V A^{2n} + \cdots \\ g_{P+2} &= 2P\,(2P-1)\, V^2 A^{2n} + \cdots \end{aligned}$$

and $d_P = 1$, $d_{P+2} = -\frac{1}{4P}$. Therefore the action starts as

$$\int dx\, A^{2n} \left[1 + \left(-3n + \frac{1}{2}\right) V^2 + \cdots\right]$$

thus reproducing the first terms in (4), and in particular the non–trivial coefficient $\left(-3n+\frac{1}{2}\right)$. In fact we can show easily that we obtain all the terms in (4). Generally we have that $g_L = k_L V^{L-P} A^{2n}$ where the coefficients k_L satisfy $k_P = 1$ and $(L+1-P)\, k_{L+1} = c_{L,P} k_L$, or $k_{L+1} = (L+P-1)\, k_L$. The recursion relation is satisfied by

$$k_L = \frac{(L+P-2)!}{(2P-2)!}.$$

Moreover the coefficients d_L are given by

$$d_{P+2q} = (-)^q \, \frac{1}{2^{2q}} \frac{(P-1)!}{(P+q-1)!q!}.$$

Putting all together we see that the action is then given by

$$\begin{aligned}\sum_{q=0}^{\infty} d_{P+2q} g_{P+2q} &= \sum_{q=0}^{\infty} (-)^q \frac{1}{2^{2q}} \frac{(P-1)!}{(P+q-1)!q!} \frac{(2P+2q-2)!}{(2P-2)!} \int dx\, V^{2q} A^{2n} \\ &= \int dx \frac{A^{2n}}{(1+V^2)^{3n-\frac{1}{2}}}.\end{aligned}$$

We have given here an example of the consistency of the construction of SW invariant actions. Many more examples can be given, all with the same basic conclusion, that SW invariance implies consistency under T–duality in the abelian case. We leave a more complete discussion to [4].

References

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